> *Decision is a risk rooted in the courage of being free.*
> *— Paul Tillich*

> *Lectures:*
>
> - *#博士生资格考试资料*

1. Introduction

> *y people will formin the line?" Queueing theory attempts to answer these questions through detailedmathematical analysis .Fundamentals of Queueing Theory*
> *show annotation*

> *lof work in the area since then. There are many valuable applications of queueing theory including traffic flow(vehicles, aircraft, people, communications), scheduling (patients in hospitals, jobson machines, programs on a computer), and facility design (banks, post offices,amusement parks, fast-food restaurants). Most real problems do not corre*
> *show annotation*

1.1 Measures of System Performance

> *1 Measures of System Performance Figure 1.1 shows a typical queueing system: Customers arrive, wait for service,receive service, and then leave the system. Some customers may leave withou*
> *show annotation*

What might one like to know about the effectiveness of a queueing system? Generally there are three types of system responses of interest:

1. 顾客的*等待时间*：waiting time that a typical customer might endure,

2. 队列或系统中*累积的顾客数*：the number of customers that may accumulate in the queue or system,
3. 服务员的*空闲时间*：idle time of the servers.

> *f the idle time of the servers. Since most queueing systems have stochastic elements,these measures are often random variables, so their probability distributions – or atleast their expected values – are sought .Regarding waiting times, there*
> *show annotation*

> *d values – are sought.Regarding waiting times, there are two types – the time a customer spends inthe queue and the total time a customer spends in the system (queue plus service) .Depending on the system being s*
> *show annotation*
>
> *等待时间分为顾客在队列中等待的时间（排队时间），以及顾客在系统中等待的总时间（排队时间和接受服务时间之和）*

Notation:

- W_q : The average waiting time of a typical customer in queue;
- W : The average waiting time in the system;
- L_q : The average number of customers in the queue;
- L : The average number of customers in the systems;

> *s denoted as W.Correspondingly, there are two customer accumulation measures – the number ofcustomers in the queue and the total number of customers in the system . The formeris of interest if we*
> *show annotation*
>
> *顾客累积数量：队列中的顾客数和系统中的顾客总输出*

> *re systemis devoid of customers. The task of the queueing analyst is generally one of two things – to determinesome measures of effectiveness for a given process or to design an "optimal" systemaccording to some*

> criterion. *To do the former, one must dete*
> [show annotation]

> *e theoptimum number of servers.* To design the waiting facility, it is necessary to haveinformation regarding the possible size of the queue . *There may also be a spacecost th*
> [show annotation]

> *, he or she may use simulation.* Ultimately,the issue generally comes down to a trade-off between better customer service andthe expense of providing more service capability, that is, determining the increase ininvestment of service for a corresponding decrease in customer delay *.4 INTRODUCTION1.2 Characteristi*
> [show annotation]

1.2 Characteristics of Queueing System

> *racteristics of Queueing Systems* A quantitative evaluation of a queueing system requires a mathematical characteri-zation of the underlying processes. In many cases, six basic characteristics providean adequate description of the system: *1. Arrival pattern of customers2*
> [show annotation]

In many cases, six basic characteristics provide an adequate description of the system:

1. 顾客的到达过程 Arrival pattern of customers
2. 服务员的服务过程 Service pattern of servers
3. 服务员的数量和服务通道的数量 Number of servers and service channels
4. 排队规则 System capacity
5. 系统容量 Queue discipline
6. 服务阶段的数量 Number of service stages

1.2.1 Arrival Pattern of Customers

> *arrivals (interarrival times).* *A common arrival process is the Poisson process, which will be described in Section 2.2* *. It is also necessary to know w*
> show annotation

> *the pattern changes with time.* *An arrivalpattern that does not change with time (i.e., the probability distribution describingthe input process is time-independent) is called a stationary arrival pattern. Onethat is not time-independent is called nonstationary.* *An example of a system witha no*
> show annotation

- *止步* (balked)

> *decide not to enter the system.* *If a customerdecides not to enter the queue upon arrival, the customer is said to have balked.* *Acustomer may enter the queue,*
> show annotation

- *中途退出* （Reneged)

> *ustomer is said to have balked.* *Acustomer may enter the queue, but after a time lose patience and decide to leave. Inthis case, the customer is said to have reneged* *. In the event that there are tw*
> show annotation

- *换队* (jockey)

> *stomer is said to have reneged.* *In the event that there are two or moreparallel waiting lines, customers may switch from one to another, that is, jockey forposition* *. These three situations are all*
> show annotation

1.2.2 Service Patterns

> *ce may also be single or batch.* *One generally thinks of one customer beingserved at a time by a given server, but there are many situations where customersmay be served simultaneously by the same server* *, such as a*

> *computer with parall*
> *show annotation*

- *状态相依服务* (state-dependent service)

> *teredand become less efficient. The situation in which service depends on the number ofcustomers waiting is referred to as state-dependent service . Service, like arrivals, canbe*
> *show annotation*
>
> *状态相依服务*

1.2.3 Number of Servers

> *tem to be fed by a single line. Thus, when specifying the number ofparallel servers, we typically assume that the servers are fed by a single line. Also,it is generally assumed that the servers operate independently of each other. 1.2.4 Queue DisciplineQueue disc*
> *show annotation*

1.2.4 Queue Discipline

> *ach other.1.2.4 Queue Discipline Queue discipline refers to the manner in which customers are selected for servicewhen a queue has formed. A common discipline in everyday*
> *show annotation*

常见的排队规则：

- *FCFS* : first come first served 先到先服务
- *LCFS* : last come first served 后到先服务
- *RSS* : random selection for service 随机服务
- *PS* : processor sharing 处理器共享
- *pooling* : 轮询（一个服务员 为多个序列的顾客提供服务，先服务第一队列的顾客吗，然后服务第二个队列的顾客，以此类推）

> *of those with lowerpriorities.* ***There are two general situations in priority disciplines, preemptive andnonpreemptive.*** *In the nonpreemptive case, the*
> show annotation
>
> *两种有限规则：抢占和非抢占*

- 非抢占情形
  具有最高优先级的顾客排在队列的最前面，但要一直等到当前正在服务的顾客的服务结束后，顾客才能接受服务，即使正在接受服务的顾客优先级更低；
- 抢占情形
  即使优先级较低的顾客已经在接受服务，也允许优先级较高的顾客在到达时立即接受服务，中断服务员对该优先级顾客的服务，只有高优先级顾客接受完成服务后，该低优先级顾客才能继续接受服务。这时又有两种情形：
  - 该顾客可以从被抢占的时刻继续接受服务；
  - 重新开始接受服务

1.2.5 System Capacity

> *until space becomes available.* ***These are referred to as finite queueingsituations; that is, there is a finite limit to the maximum system size.*** *A queue withlimited waiting room*
> show annotation

1.2.6 Stages of Service

> *eedback may occur (Figure 1.3).* ***Recycling is commonin manufacturing processes, where quality control inspections are performed aftercertain stages, and parts that do not meet quality standards are sent back for repro-cessing.*** *Similarly, a telecommunications*
> show annotation

1.2.7 Notation

> *tem with feedback.1.2.7 Notation* ***As shorthand for describing queueing processes, a notation has evolved, due for themost part to Kendall (1953),***

> which is now rather standard throughout the queueingliterature . A queueing process is describe
> show annotation

A queueing process described by a series of symbols and slashes (斜线) $A/B/X/Y/Z$:

- A : denotes the inter-arrival time distribution (到达时间间隔分布)
- B : service time distribution (服务时间分布)

表 1.1 排队系统表示法 $A/B/X/Y/Z$

特征	符号	说明
到达时间间隔分布（A） 服务时间分布（B）	M	指数分布
	D	确定性分布
	E_k	k 阶埃尔朗分布（$k=1,2,\cdots$）
	H_k	k 阶超指数分布（$k=1,2,\cdots$）
	PH	阶段型分布
	G	一般分布
并行服务员数（X）	$1,2,\cdots,\infty$	
系统容量（Y）	$1,2,\cdots,\infty$	
排队规则（Z）	FCFS	先到先服务
	LCFS	后到先服务
	RSS	随机服务
	PR	优先级
	GD	一般规则

- 例子

> *$M/D/2/\infty/\mathrm{FCFS}$ （或 $M/D/2$）表示这样一个排队系统:*
>
> - *到达时间服从指数分布*
> - *服务时间是定长的*
> - *有两个并行的服务员*
> - *系统容量无限（即允许进入系统的顾客数没有限制）*
> - *排队规则是先到先服务*

通常，如果系统容量没有限制，即 $Y=\infty$），则省略系统容量的符号；如果排队规则是先到先服务（Z=FCFS），则省略排队规则的符号。因此 $M/D/2/\infty/\mathrm{FCFS}$ 和 $M/D/2$ 表达的含义相同。

> sed for the Erlangdistribution. Rather, M is used, standing for the Markovian or memoryless propertyof the exponential (described in Section 2.1) . Second, the symbol G represent
> show annotation

1.2.8 Model Selection

> e there are c checkoutcounters. If customers choose a checkout counter on a purely random basis (withoutregard to the queue length in front of each counter) and never switch lines (nojockeying), then we have c independent single-server models. If, instead, there is asingle w
> show annotation

> ndependent single-serverqueues. As jockeying is rather easy to accomplish in supermarkets, the c-servermodel with one queue may be more appropriate and realistic than c independentsingle-server models, which one might have been tempted to choose initially prior togiving much thought to the process. 1.3 The Experience of WaitingThi
> show annotation

1.3 The Experience of Waiting

> proved in a number ofother ways. This section summarizes several principles, proposed by Maister (1984),related to the experience or psychology of waiting. The reader can likely relate to
> show annotation

1. Unoccupied time feels longer than occupied time.
2. Pre-process wait feels longer than in-process wait.
3. Anxiety makes waiting seem longer.
4. Uncertain waits are longer than known, finite waits.
5. Unexplained waits are longer than explained waits.
6. Unfair waits are longer than equitable waits.
7. Longer waits are tolerable for more valuable service.
8. Solo waits feel longer than group waits.

1.4 Little's Law

> *han group waits.1.4 Little's Law A fundamental relationship that is used extensively in queueing theory and throughoutthis text is Little's law. Little's law provides a relatio*
> show annotation

Little's law provides a relationship between three fundamental quantities: The average rate λ that customers arrive to a system, the average time W that a customer spends in the system, and the average number L of customers in the system.

$$L = \lambda W$$

> *g-run average rate of arrivals. The second limit W is thelong-run average time spent in the system per customer. The third limit L is thelong-run average number of customers in the system .Theorem 1.1 [Little's law] If t*
> show annotation

> *mber of customers in the system. Theorem 1.1 [Little's law] If the limits λ and W in (1.1) exist and are finite, thenthe limit L exists and L = λW.12 INTRODUCTIONLittle's L*
> show annotation

> *2011) in aretrospective article. Before giving examples, we make some general remarks about Little's law. First,Theorem 1.1 is a statemen*
> show annotation

在给出例子之前，需要对 Little 法则进行一些一般性的解释说明：

1. 定理用于计算长期平均值，即式中的 L, λ, W 被定义为无穷极限；
2. 定理要求 λ 和 W 有极限存在，这就排除了当时间无限增长时系统的指标无界的情况；
3. 定理没有要求必须存在一个队列，但要求存在一个系统，顾客可以到达和离开该系统。

1.4.1 Geometric Illustration of Little's Law

> *ric Illustration of Little's Law* *We now give a geometric "proof" of Little's law. This is not a rigorous proof, butrather a rough argument showing the main ideas behind Little's law.* *Full technicalproofs can be fou*
> show annotation

2. Review of Stochastic Processes

> *R 2REVIEW OF STOCHASTICPROCESSES* *This chapter provides an overview of key concepts in stochastic processes usedthroughout this text* *. Topics include the exponential*
> show annotation

2.1 The Exponential Distribution

> *2.1 The Exponential Distribution* *In queueing theory, the exponential distribution is often used to model the time until aparticular event occurs* *– for example, the time until t*
> show annotation

> *of theexponential distribution.* *We will see (Section 2.2) that the exponential distribution isclosely connected with the Poisson process, another widely used model in queueingtheory.* *The exponential distribution is*
> show annotation

定义：服从指数分布的随机变量是连续型随机变量，其 *概率密度函数* pdf 为：

$$f(t) = \lambda e^{-\lambda t}$$

服从指数分布的随机变量 T 的 *累积分布函数* (cumulative distribution function, CDF)、*互补累积分布函数* (complementary cumulative distribution function, CCDF)、期望和方差可以通过其概率密度函数求得，分别表示为：

$$\begin{aligned} F(t) &\equiv \Pr\{T \le t\} = 1 - e^{-\lambda t} \\ \bar{F}(t) &\equiv \Pr\{T > t\} = e^{-\lambda t} \\ \mathrm{E}[T] &= \tfrac{1}{\lambda}, \quad \mathrm{Var}[T] = \tfrac{1}{\lambda^2} \end{aligned}$$

> scussed in Chapters 3, 4, and 5. Definition 2.1 An exponential random variable is a continuous random variablewith probability density function (PDF): f(t) = λe−λt (t ≥0),where λ > 0
>
> show annotation

$$Pr\{T > t + s | T > s\} = Pr\{T > t\} \quad (s, t \geq 0)$$

> unt of time spentwaiting so far. Theorem 2.1 An exponential random variable has the memoryless property .Proof: The proof is fairly stra
>
> show annotation

> .THE EXPONENTIAL DISTRIBUTION 37 Note that $Pr\{T > t + s, T > s\} = Pr\{T > t + s\}$ (if T is bigger than $t + s$, then it is also bigger than s). We now consider an example of a
>
> show annotation

> t can be found in manytextbooks. Theorem 2.4 Let $T_1, \cdots, T_n$ be independent exponential random variables with rates $\lambda_1, \cdots, \lambda_n$, respectively. Then Pr{Ti = min{T1,...,Tn}}= λiλ1 +
>
> show annotation

$$Pr\{T_i = \min\{T_1, \cdots, T_n\}\} = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_n}$$

> stated more formally as follows. Theorem 2.5 Let T1,...,Tn be independent exponential random variables withrates λ1,...,λn, and let T = min{T1,...,Tn}. Then the event {Ti = T} isindependent of T .2.2 The Poisson ProcessThe Pois
>
> show annotation

2.2 The Poisson Process

> ent of T.2.2 The Poisson Process The Poisson process is a common process for modeling arrivals to a queueing system.Intuitively, the process can be thought of describing events that occur “randomly” intime. The concept of randomness will b
>
> show annotation

- *Stochastic process* (随机过程) $\{N(t), t \geq 0\}$: a collation of random variables indexed by time.
- *Counting process* (记数过程) : a stochastic process in which $N(t)$ takes on nonnegative integer values and is nondecreasing in time.

> *s and is nondecreasing in time. A counting process typicallyrepresents the cumulative number of events that have occurred by time t. With thesepreliminaries, we give a definition of the Poisson process .Definition 2.3 A Poisson proces*
> show annotation

> *finition of the Poisson process. Definition 2.3 A Poisson process with rate $\lambda > 0$ is a counting process $N(t)$ with the following properties: 1. N(0) = 0.2. Pr{1 event betwee*
> show annotation

到达速率为 $\lambda > 0$ 的泊松过程，满足以下性：

1. $N(0) = 0$
2. $Pr\{1 \text{ event between} t \text{ and } t + \Delta t\} = \lambda\,\Delta\,t + o(\Delta t).$
3. $Pr\{2 \text{ or more events between} t \text{ and } t + \Delta t\} = o(\Delta t).$
4. The numbers of events in nonoverlapping intervals are statistically independent; that is, the process has independent increments (独立增量过程).

> *on of a Poisson random variable. Definition 2.4 A Poisson random variable is a discrete random variable with prob-ability mass function pn = e−A Ann! (n = 0,1,2,...),wh*
> show annotation

泊松随机变量是离散随机变量，其概率质量函数为：

$$p^n = e^{-A}\frac{A^n}{n!}, \quad n = 0, 1, 2, \cdots$$

其中，A 是大于 0 的常熟，泊松随机变量 X 的期望和方差分别为

$$\mathbb{E}[X] = A, Var[X] = A$$

tical induction (Problem 2.2). Poisson processes have a number of interesting additional properties, which arestated in the following theorems. The first result is that a Poisson process hasstationary increments. This means that the distribution of the number of events ina given time interval (i.e., an increment) depends on the length of the interval butdoes not depend on the absolute location of the interval in time . For example, thenumber of even

show annotation

泊松过程具有平稳增量性，即在一个给定的时间区间内发生的事件数（即增量）的分布仅取决于改区间的长度，与时间区间内的绝对位置无关。

have occurred on theinterval. The notion that event times are “completely random” comes from the factthat they are uniformly distributed in time. However, we must be precise about whatwe mean by “event times.” Specifically, we must distinguish between ordered and un-ordered event times. To illustrate the difference, i

show annotation

4 REVIEW OF STOCHASTIC PROCESSES One important consequence of the uniform property of the Poisson process is thatthe outcomes of random observations of a stochastic process X(t) have the sameprobabilities as if the scans were taken at Poisson-selected points . When X(t) is aqueue, this prop

show annotation

processes.THE POISSON PROCESS 45 Theorem 2.10 (Splitting) Let N(t) be a Poisson process w

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分流

are independent, for all i 6= j. Theorem 2.11 (Superposition) Let N1(t),...,Nn(t) be independ

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> *汇合*

2.2.1 Generalizations of the Poisson Process

> *reater detail later in the text. The first generalization considered is a nonhomogeneous Poisson process (NHPP).A NHPP can be thought of as a Poisson process where the arrival rate λ is replaced bya time-dependent function λ(t) . This type of situation is quit*
> [show annotation]

> *6 REVIEW OF STOCHASTIC PROCESSES Definition 2.5 A nonhomogeneous (or nonstationary) Poisson process is a Poissonprocess (Definition 2.3) in which assumption 2 is replaced by the following :Pr{1 arrival between t and t +*
> [show annotation]

非齐次泊松过程 的定义为：

$$Pr\{1 \text{ arrival between } t \text{ and } t+\Delta t\} = \lambda(t)\Delta t + o(\Delta t)$$

从定义中可以发现，其到达速率 $\lambda(t)$ 在一天中随时间 t 变化。
Note：当非齐次泊松过程的到达速率 $\lambda(t)$ 是常数时，可将非齐次泊松过程视为标准泊松过程。

Theorem 2.12 For a non-homogeneous Poisson process $N(t)$ with mean event rate $\lambda(t)$, the number of events in a time interval $(s, t]$ is a Poisson random variable with mean $m(t) - m(s)$, where

$$m(t) \equiv \int_0^t \lambda(u)\,\mathrm{u}$$

The difference $m(t) - m(s)$ can be computed by integrating $\lambda(u)$ @a

> *) −m(s), wherem(t) ≡∫ t0λ(u) du. The function $m(t)$ is sometimes called the mean value function It represents the cumulative expected number of events by time t. The standard Poisson process is*
> [show annotation]

- *CPP* : compound Poisson process 复合泊松分布

> *vals is 1 −∑99n=0 e−190 190nn! .* *The next generalization is a compound Poisson process (CPP).* *A CPP is like aPoisson process b*
> *show annotation*

复合泊松分布类似于标准泊松分布，但在复合泊松分布过程中，事件按批次发生。

> *e have the following definition.* *Definition 2.6 Let $M(t)$ be a Poisson process, and let Y_n be an i.i.d. sequence of strictly positive integer random variables that are independent of $M(t)$. Then* *N(t) ≡M(t)∑n=1Yn.is a compound P*
> *show annotation*

Formulation:

$$N(t) \equiv \sum_{n=1}^{M(t)} Y_n$$

Example : 将一辆公交车视为一个批次，车上所有乘客在同一批次到达下一个站点。*批次数*（如到达站点的公交车数）服从泊松分布，则 *到达的顾客数*（如公交车上的乘客数）服从 *复合泊松分布*。其中 $M(t)$ 表示时刻 t 前到达的公交车数，Y_n 表示第 n 亮公交车上的乘客数，$N(t)$ 表示时刻 t 前到达的总乘客数。

> *f people who have arrived by t.* *For a given value of t,N(t) is a compound Poisson random variable, since the number of terms in the sumis random and follows a Poisson distribution (and this number is independent of Yn)* *.Compared to a standard Poisson*
> *show annotation*

与标准泊松过程相比，复合泊松过程具有独立且平稳的增量，但不具有 *有序性*，即复合泊松过程是将 Sec 2.2 定义中的性质（2）和性质（3）替换为以下性质的泊松过程：

$$Pr i \text{ arrivals in } (t, t+\Delta t] = \lambda i \, \Delta \, t + o(\Delta t) \quad (i = 1, 2, \dots),$$

其中，$\lambda_i \equiv c_i \lambda$ 是大小为 i 的批次的 *有效到达速率*。

> *arrival rate of size-i batches. For a CPP, it is relatively straightforward to derive the mean and variance of N(t)(e.g., Ross, 2014): E[N(t)] = λtE[Yn], and Var[N(t)]*
> [show annotation]

Mean and variance of $N(t)$:

$$\mathbb{E}[N(t)] = \lambda t \mathbb{E}[Y_n]$$
$$\mathrm{Var}[N(t)] = \lambda t \mathbb{E}[Y_n^2]$$

- **Renewal Process**：更新过程
  更新过程是 *非负独立同分布* 随机变量的集合，这些随机变量表示连续发生的事件之间的事件间隔。

> *1/9) + (33/3!)(1/216)] .= 0.076. The Poisson process is special case of a larger class of problems called renewalprocesses. A renewal process arises from a sequence of nonnegative IID randomvariables denoting times between successive events. For a Poisson process, theinter*
> [show annotation]

> *imes between successive events. For a Poisson process, theinter-event times are exponential, but for a renewal process, they follow an arbitrarydistribution G. Many of the properties that we*
> [show annotation]
>
> *泊松过程中，事件发生的事件间隔服从指数分布；但在更新过程中，事件发生的时间间隔服从任意分布 G*

> *stringent, this is not the case. A strong argument in favor of exponential inputs is the one that often occurs in thecontext of reliability. It is the result of the well-kno*
> [show annotation]

> *om the theoryof extreme values. Here, the exponential appears quite frequently as the limitingdistribution of the (normalized) first-order statistic of random samples drawn fromcontinuous populations (see Problem 1.10 for one such example). There is also anadditional argu*

> show annotation
>
> *通过观察从连续总体中抽取的随机样本可以发现，随机样本（归一化后的）第一顺序统计量的极限分布通常为指数分布。*

> *omes out of information theory. It is that the exponentialdistribution is the one that provides the least information, where information content DISCRETE-TIME MARKOV CHAINS 49or*
> show annotation
>
> *指数分布是提供信息量最少的分布*

指数分布 $f(x)$ 的信息量或负熵被定义为：

$$\int f(x)\log f(x)\,\mathrm{d}\,x$$

2.3 Discrete-Time Markov Chains

> *.2.3 Discrete-Time Markov Chains In this section, we consider a class of models in which the system transitions amonga discrete set of states at various points in time . Figure 2.3 shows an example sy*
> show annotation

> *If or to state 2, and so forth. Inqueueing applications, the system state is often defined as the number of customers inthe system, in which case the state space is the set of nonnegative integers 0,1,2,... .0132Figure 2.3 Markov chain wit*
> show annotation

马尔可夫链的基本假设是具有 *马尔可夫性*，即

$$Pr\{X_{n+1}=j|X_0=i_0,X_1=i_1,\cdots,X_n=i_n\}=Pr\{X_{n+1}=j|X_n=i_n\}$$

> *Xn = in}= Pr{Xn+1 = j|Xn = in}.I Intuitively, the Markov property states that if the “present” state of the system (Xn) is known, then the “future” (X_{n+1}) is independent of the “past” $(X_0,...X_{n-1})$. Inother words, in order to cha*

> show annotation
>
> *这表明如果系统当前的状态是已知的，那么未来的状态与过去的状态无关。*

> *relevant given thepresent state. The conditional probabilities $Pr\{Xn+1=j|Xn=i\}$ are called the single-step transition probabilities or just the transition probabilities. Often these probabilitiesare ass*
> show annotation

> *arkov chain (e.g., Section 6.3). For a Markov chain, one may be interested in the m-step transition probabilities,defined as the probability of being in state j exactly m steps after being in state i .More precisely, the m-step tran*
> show annotation

> *= i},which is independent of n. Let $P^{(m)}$ be the matrix formed by the elements $p_{ij}^{(m)}$ .From the basic laws of probability , it can be shown thatP(m) = P ·*
> show annotation

- *CK* equation: 查普曼-科尔莫戈罗夫方程 (Chapman-Kolmogorov equation)

> *tep matrix P by itself m times. This is the matrix equivalent of thewell-known Chapman Kolmogorov (CK) equations for this Markov process .A similar argument can be used*
> show annotation

2.3.1 Properties of Markov Chains

> *associated with Markov chains. State j is accessiblefrom state $i(i \to j)$ if there exists an n ≥ 0 such that $p_{ij}^{(n)} > 0$. That is, there is some path from i to j with nonzero probability . Two states i and j communicate*
> show annotation

1. 如果存在 $n \geq 0$，使得 $p_{ij}^{(n)} > 0$， 则系统可以从状态 i 到达状态 j ($i \to j$)，即存在从状态 i 到状态 j 的非零概率路径。
2. *连通* (cummunicate)：如果 $i \to j$ 且 $j \to i$，则称状态 i 和状态 j 是连通的 $i \leftrightarrow j$。

3. *等价类* (communication class)：性质2 将马尔可夫链的状态划分为多个互不相交的子集，被称为等价类。
    - 一个等价类中的所有状态都是连通的，并且该等价类中的状态不与任何其他等价类中的状态想连通。
4. *不可约的* (irreducible)：如果一个马尔可夫链的所有状态都是连通的，系统可以从任意状态到达任意其他状态，则称为不可约，否则，称为 *可约的* (reducible)
5. *常返的* (recurrent) ： 从状态 j 除法，返回状态 j 的概率为 1；否则称状态 j 为 *瞬时的* (transient)

> *erwise, the state is transient. More precisely, let $f_{jj}^{(n)}$ be the probability that a chain starting in state j returns for the first time to j in n transitions. The probabilitythat the chain ever returns to j is fjj =∞∑n=1f(n)jj .†A state j is*
> show annotation

设马尔可夫链从状态 j 除法，转移 n 步之后首次返回状态 j 的概率为 $f_{jj}^{(n)}$，则该链返回状态 j 的状态之和为

$$f_{jj} = \sum_{n=1}^{\infty} f_{jj}^{(n)}$$

如果 $f_{jj} = 1$，则状态 j 是常返的；如果 $f_{jj} < 1$ ，则状态 j 是瞬时的。当 $f_{jj} = 0$ 时，

$$m_{jj} = \sum_{i=1}^{\infty} n f_{jj}^{(n)}$$

表示 *平均回转时间* (mean recurrence time)。此时，又有以下分类：
7. *正常返的* (positive recurrent)： $m_{jj} < \infty$
8. 零常返的* (positive recurrent)： $m_{jj} = \infty$

> *null recurrence and transience. The period of astate j is the greatest common divisor of integers m such that p(m)jj > 0. A state withperiod 1 is said to be aperiodic .EXAMPLE 2.7Consider the followi*
> show annotation

状态 j 的 *周期* (period) 是满足 $p_{jj}^{(m)} > 0$ 的所有正整数 m 的最大公约数。周期为 1 的状态是 *非周期的* (aperiodic)

2.3.2 Long-Run Behavior

> *ARKOV CHAINS 53In this example, the limiting matrix has the property that the rows are the same. Thismeans that, a long time into the future, the probability of being in a particular statedoes not depend on the starting state. For example, if the system star*
> *show annotation*

> *if thesystem starts in state 1. This particular behavior does not hold for all Markov chains. First, it is not alwaysthe case*
> *show annotation*

Note : 但并不是所有的马尔可夫链都有这种特殊的行为。因为

1. $n \to \infty$ 时，P^n 并非总是收敛的；
2. 如果 p^n 收敛，矩阵每行的元素可能并不相同。
   由此，可以引出马尔可夫链的长期行为相关的 3 个概念：

- 极限分布 (limiting distributions)
- 平稳分布 (stationary distributions)
- 遍历性 (ergodicity)

> *, the rows may notbe identical. This motivates discussion of three related concepts having to do withlong-run behavior, limiting distributions, stationary distributions, and ergodicity. We start by defining the limitin*
> *show annotation*

> *y distributions, and ergodicity. We start by defining the limiting probabilities of a Markov chain as πj ≡ limn→∞p(n)ij . (2.14)This d*
> *show annotation*

马尔可夫链的极限概率为：

$$\pi_j \equiv \lim_{n\to\infty} p_{ij}^{(n)}$$

> *∑k[limm→∞p(m−1)ik]pkj= ∑kπkpkj. The step of rearranging the brackets requires switching a limit and a sum. If theMarkov chain has an infinite number of states (i.e., P is an infinite-dimensionalmatrix), then this step must be justified more carefully (e.g., see Harchol-Balter,2013)*
> [show annotation]

> *tence of a limitingdistribution. Theorem 2.13 An irreducible and positive recurrent discrete-time Markov chainhas a unique solution to the stationary equations π = πP and ∑jπj = 1, (2.16)namel*
> [show annotation]

定理 对于一个不可约且正常返的离散时间马尔可夫链，一下平稳方程组有唯一解：

$$\pi = \pi P \quad \text{and} \quad \sum_j \pi_j = 1$$

即 $\pi_j = 1/m_{jj}$，如果该马尔可夫链是非周期性的，则极限分布存在并且与平稳分布相同。

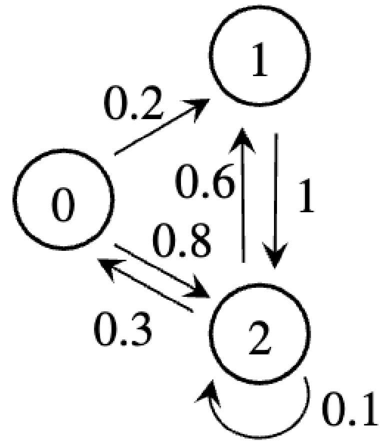
> *to the stationary distribution. By this theorem, there are two main ways to interpret the value of π_j . The firstinterpretation is tha*
> [show annotation]

> *to interpret the value of πj. T he firstinterpretation is that πj is the the long-run fraction of time spent in state j. This comesfrom the fact that πj = 1/mjj . Recall that mjj is the mean ti*
> [show annotation]

> *nstate j (from renewal theory). The second interpretation is that πj is the probabilityof being in state j a long time from now (more precisely, πj is a limiti*
> [show annotation]

Example

$$
\boldsymbol{P} = \begin{pmatrix} 0 & .2 & .8 \\ 0 & 0 & 1 \\ .3 & .6 & .1 \end{pmatrix}.
$$



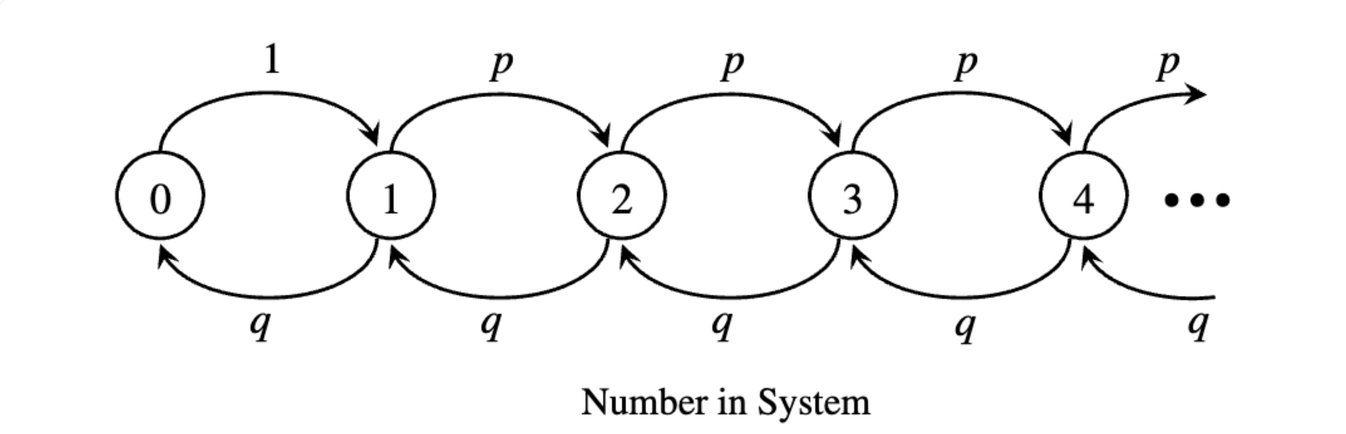
因为所有的状态都是连通的，所以该马尔可夫链是 *不可约* 的，且是 *正常返的*。**具有有限状态数的不可约马尔可夫链一定是正常返的**。带入上式中的平稳方程组可得：

$$
\begin{cases}
\pi_0 = 0.3\pi_2 \\
\pi_1 = 0.2\pi_0 + 0.6\pi_2 \\
\pi_2 = 0.8\pi_0 + \pi_1 + 0.1\pi_2 \\
\pi_0 + \pi_1 + \pi_2 = 1
\end{cases}
$$

```language-python
import numpy as np

p = np.array([[0, 0.2, 0.8],[0,0,1],[0.3,0.6,0.1]])
np.linalg.matrix_power(p,10)
np.linalg.matrix_power(p,40)
```

Results:

```language-results
array([[0.17133537, 0.36886623, 0.4597984 ],
       [0.18579266, 0.39428656, 0.41992079],
       [0.12597624, 0.289111  , 0.58491276]])



array([[0.15312356, 0.3368443 , 0.51003214],
       [0.15317288, 0.33693102, 0.50989611],
       [0.15296883, 0.33657224, 0.51045893]])
```



状态转移图

> er in System1 2 3 41 p p p qp... This chain is the embedded discrete-time Markov chain for the $M/M/1$queue (see Example 2.15), where the state of the system is measured onlywhen an arrival or departure occurs (i.e., at discrete points in ti
> show annotation

> n probability matrixP =(0 11 0). The chain is irreducible and positive recurrent, so it has a unique solution tothe stationary equations. We can solve (2.16) to get the
> show annotation

> c chain tobe positive recurrent. Theorem 2.14 An irreducible, aperiodic chain is positive recurrent if there exists anonnegative solution of the system ∞∑j=0pijxj ≤xi −1 (i 6= 0)such t
> show annotation

2.3.3 Ergodicity

> ∞∑j=0p0jxj < ∞.2.3.3 Ergodicity Closely associated with the concepts of limiting and stationary distributions is theidea of ergodicity, which has to do with the information contained in one infinitelylong sample path of a process (e.g., Papoulis, 1991). Ergodici
> show annotation

> *遍历性与极限分布和平稳分布密切相关，并且与包含在某个过程中的无限长样本路径中的信息有关。*

> *process (e.g., Papoulis, 1991). Ergodicity is important in that itdeals with the problem of determining measures of a stochastic process X(t) from asingle realization, as is often done in analyzing simulation output . X(t) is ergodic inthe most gen*
> [show annotation]

遍历性可以帮助我们基于过程的单个样本实现来确定随机过程 $X(t)$的统计指标。如果 $X(t)$ 的所有指标都可以基于过程的单个样本实现 $X_0(t)$ 来确定或较为准确地估算，那么 $X(t)$ 在一般意义上是 *遍历的。*

> *lization X0(t) of the process. S ince statistical measures of theprocess are usually expressed as time averages, this is often stated as follows: X(t)is ergodic if time averages equal ensemble averages. (Here, the time parameter t isc*
> [show annotation]
>
> *随机过程平均值等于集合平均值*

> *average versus ensemble average. For a nonstationary process, the ensemble average m(t) might be different atdifferent values of t. For example, if a queueing system starts in an empty state, thenthe ensemble average at t = 0 will be different than the ensemble average at somelarge value of t, where the system is in steady state . Nevertheless, we might imagine*
> [show annotation]

> *nt) ifx = limt→∞m(t) < ∞. (2.18) That is, the ensemble average m(t) converges to a limit as $t \to \infty$ and this limitingvalue equals the time average . For a stationary process, the*
> [show annotation]

> *sted in fully ergodic processes. We now discuss the link between a limiting distribution, a stationary distribution, and ergodicity. Consider a DTMC that is*

> *irreducible and positive recurrent. Such a chain has a unique stationary distribution $\{\pi_i\}$ by Theorem 2.13* . *Furthermore,πi is the long-run*
> show annotation

2.4 Continuous Time Markov Chains

2.4.1 Embedded Markov Chains

A (time-homogeneous) continuous-time Markov chain (CTMC) is a stochastic pro-
cess $\{X(t), t \geq 0\}$ with a countable state space, such that:

1. Each time the process enters state i, it remains in that state for a period of time that is exponentially distributed with rate v_i (independent of the past).
2. When the process departs state i, it goes to state $j \neq i$ with probability p_{ij} (independent of the past).

Note : *连续时间（齐次）马尔可夫*（CTMC）从一个状态转移到另一个状态的过程与*离散时间马尔可夫链*（DTMC）相似，但是 CTMC 在每个状态停留的时间是*连续型指数随机变量*。由*转移矩阵* $\{p_{ij}\}$ 定义的 DTMC 被称为嵌入离散时间马尔可夫链（embedded discrete-time Markov chain）。

> *e back to itself are notallowed. In continuous time, the Markov property can be stated as* *Pr{X(t + s) = j|X(t) = i,X(u),0*
> show annotation

在连续时间上， 马尔可夫性可以表述为：

$$Pr\{X(t+s)=j|X(t)=i, X(u), 0 \leq u < t\} = Pr\{X(t+s)=j|X(t)=i\}$$

2.4.1 Embedded Markov Chains

> *S 652.4.1 Embedded Markov Chains In many of the situations in this text requiring the use of a continuous-time queueingmodel, we can often get satisfactory results by looking at the state of the systemonly at certain selected times* *, leading to an embedded discret*
> show annotation

在许多场景中，要求使用连续时间排队模型，在这些场景下，通常需要在特定时间点观察系统的状态，由此，引入 *嵌入离散时间马尔可夫链* 来解决此类问题。

> *owing the nth state transition. Asdiscussed previously, if the system is in state i ≥1, the next event is an arrivalwith probability λ/(λ+μ) and a service completion with probability μ/(λ+μ). When i = 0 (empty system), the n*
> *show annotation*
>
> *定理2.4，只有两个事件*

> *dded discrete-timeMarkov chain. More generally, there are some continuous-time processes that arenot CTMCs but still have embedded discrete-time Markov chains. For instance,processes associate*
> *show annotation*

2.4.2 Chapman-Kolmogoroc Equations

> *4.2 Chapman–Kolmogorov Equations For a DTMC, we were able to determine the n-step transition probabilities via theChapman–Kolmogorov equations. From this, we obtained an expli*
> *show annotation*

> *ystem of differential equations. Theorem 2.15 Let $p_i(t)$ be the probability that the system is in state i at time t, let $p(t)$ be the vector $(p_0(t), p_1(t), \cdots)$, and let p′(t) be the vector of its derivatives.Then p′(t) = p(t)Q. (2.21)66 REVIEW O*
> *show annotation*

定理 设 $p_i(t)$ 为系统在时刻 t 处于状态 i 的概率， $p(t)$ 表示向量 $(p_0(t), p_1(t), \cdots)$，且 $p'(t)$ 为 $p(t)$ 的导数，则：

$$p'(t) = p(t)Q$$

其分量形式为：

$$p_j'(t) = -v_j p_j(t) + \sum_{r \neq j} p_r(t) q_{rj}$$

2.4.3 Long-Run Behavior

> *lsewhere.2.4.3 Long-Run Behavior The same concepts of stationarity and steady state apply for the continuous-timecase, with t replacing n in the limiting process. For example, analogous to (2.14*
> [show annotation]

> *stated in the followingtheorem. Theorem 2.16 For a continuous-time Markov chain, if the embedded discrete-timechain is irreducible and positive recurrent, then there is a unique solution to the PROBLEMS 69stationary equations0*
> [show annotation]

定理 2.16 对于一个连续时间马尔可夫链，如果其对应的嵌入离散时间马尔可夫链是不可约且正常返的，那么以下平稳方程组存在唯一解：

$$
\begin{cases} 0 = pQ \\ \sum_j p_j = 1 \end{cases}
$$

其中，0 是零向量。

> *to 0,then (2.21) becomes 0 = pQ. Compared to a discrete-time chain (Theorem 2.13), aperiodicity is not required forthe limiting distribution to exist in a continuous-time Markov chain . This is becausethe times betwe*
> [show annotation]

3. Simple Markovian Queueing Models

> *theoryof birth death processes. Recall that a birth death process is a specific type ofcontinuous-time Markov chain whose structure leads to a straightforward solutionfor the steady-state probabilities $\{p_n\}$. Examples of queues that can be*
> [show annotation]

> *dent arrival and service rates. Webegin with the general theory of birth death process. Then we apply these results toobtain measures of effectiveness for*

> the queueing systems given above .3.1 Birth Death ProcessesA birt
> show annotation

3.1 Birth-Death Processes

> above.3.1 Birth Death Processes A birth death process consists of a set of states {0, 1, 2, . . . }, typically denoting the“population" of some system .
> State transitions occur as uni
> show annotation

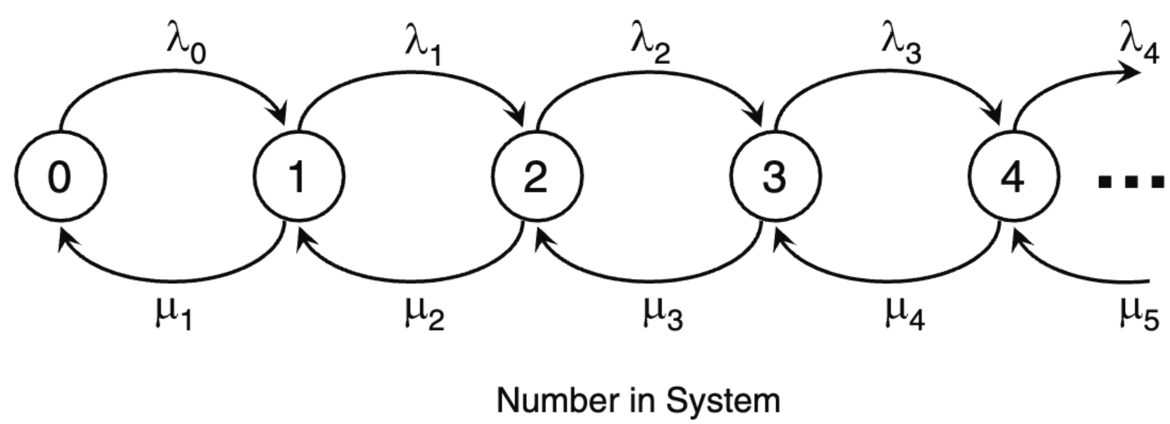


Figure 3.1 Rate transition diagram for a birth–death process.

从 Sec 2.4.3 定理2.16 中可知，该系统存在一个解，基于 $0 = pQ$，且当 λ_n 和 μ_n 有一定的条件限制时，可以求得该解。对于生灭过程，向量矩阵的分量形式为：

$$
\begin{aligned}
0 &= -(\lambda_n + \mu_n)p_n + \lambda_{n-1}p_{n-1} + \mu_{n+1}p_{n+1} \\
0 &= -\lambda_0 p_0 + \mu_1 p_1
\end{aligned}
$$

也可以写为（流量平衡，flow balance）：

$$
\begin{aligned}
(\lambda_n + \mu_n)p_n &= \lambda_{n-1}p_{n-1} + \mu_{n+1}p_{n+1} \\
0 &= -\lambda_0 p_0 + \mu_1 p_1
\end{aligned}
$$

第一个式子左边表示从状态 n 转移 *出去* 的速率，右边表示从其他状态 *进入* 状态 n 的速率。

> 1pn+1 (n ≥ 1), (3.1)λ0p0 = μ1p1. These equations can also be obtained using the concept of flow balance. The basicidea is this: In steady state, the rate of transitions out of a given state must equal therate of transitions into that

> state. As we illustrate in a moment, t
> show annotation

求解可以得到：

$$p_n = \frac{\lambda_{n-1}\lambda_{n-1}\cdots\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_1}p_0 = p_0\prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i}, \quad n \geq 1$$

进而可以求解得到：

$$p_o = \left(1 + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i}\right)^{-1}$$

> ∏i=1λi−1μi)−1. (3.4)From (3.4), we see that a necessary and sufficient condition for the existence of asteady-state solution is the convergence of the infinite serie s1 + ∞∑n=1n∏i=1λi−1μi.As we will
> show annotation
>
> 可以发现，稳态存在解的充要条件是无穷级数收敛。

可以发现，稳态存在解的充要条件是以下无穷级数收敛：

$$1 + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\lambda_{i-1}}{\mu_i}$$

>) and (3.4)starting from (3.1). The equations in (3.1) are called global balance equations, since they equate the total mean flow into each state with the total mean flow out of that state .Yet there is an alternate set o
> show annotation
>
> 整体平衡方程

两种不同思路：

- 上述方法称为 *整体平衡方程* (global balance equation)
- *局部平衡方程* (detailed balance equation) 。正如稳态时 *流入和流出* 一个状态的平均流量必须相等，稳态时向左和向右通过分界线的平均流量也必须相

等。如下图

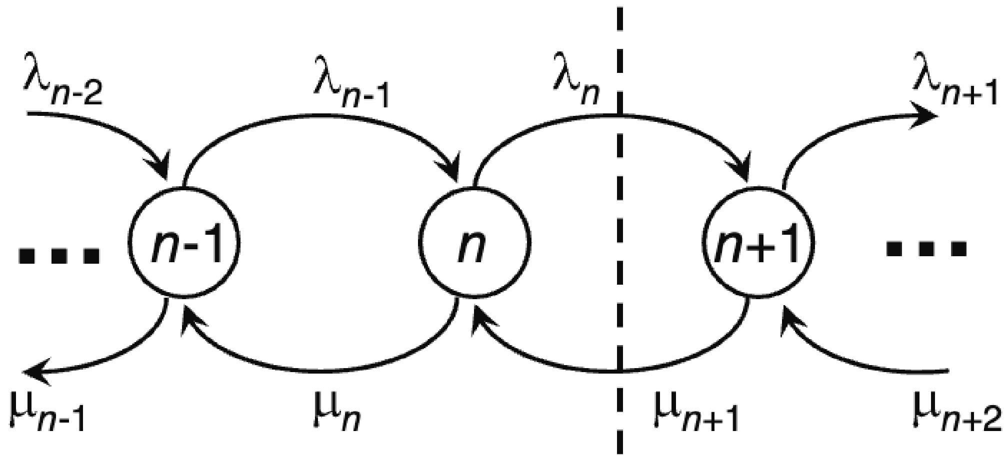


Figure 3.2 Flow balance between states.

we can know:

$$\lambda_{n-1}p_{n-1} = \mu_n p_n$$
$$p_n = \frac{\lambda_{n-1}}{\mu_n}p_{n-1}$$

> *1 and n, as shown in Figure 3.2. Just as mean flows into and out of a state must be equal in steady state, so alsomean flows across the barrier must be equal in steady state. This can be seen asfollows: If*
> [show annotation]

- 可逆性 (reversibility)

> *global balance equations (3.1). It is not true for all Markov chains that the mean flows between two states areequal. The equating of these adjacent*
> [show annotation]

> *ws between two states areequal. The equating of these adjacent flows relates to something called reversibility,a concept that becomes particularly useful later in our work on queueing networks (see Section 5.1.1 and also Sect*
> [show annotation]

> *tates candirectly communicate. F or more general Markovian models, this is not necessarilytrue. However, for all Markovian models, equating the total flow out of a state withthe total flow into the state always yields the global balance equations , from whichthe {pn} can be dete*
> [show annotation]

Note : 并非所有马尔可夫链的两个状态之间的 *平均流量* 都想等 （与 *可逆性* 有关）。但是，对于所有马尔可夫过程，流出一个状态的总流量与流入该状态的总流量想等，所以一定可以得到整体平衡方程，从而可以求得 $\{p_n\}$。

3.2 Single Server Queues ($M/M/1$)

更多知识查看：

> Notes for Queueing Theory
>
> # 13 排队论
>
> ## 1. 背景知识
>
> ### 1.1 Notation
>
> - 肯德尔记号 (Kendall)：输入分布/输出分布/并联服务台数（$X/Y/Z$）
>
> 1971 年，国际排队符号标准会上扩展至六项，记为 ($X/Y/Z/A/B/C$)：
>
> ```markdown
> 输入分布/输出分布/并联服务台数/系统容量（队长）/系统状态（顾客源数）/服务规则
> ```
>
> e.g. $M/M/1/\infty/\infty/FCFS$
>
> - 泊松流
>
> $$P_n(t) = \frac{(\lambda t)^n}{n!}e^{-\lambda t}$$

- *负指数分布*

  PDF:

$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

  CDF:

$$F_T(t) = \begin{cases} 1 - e^{-\lambda t}, & t \ge 0 \\ 0, & t < 0 \end{cases}$$

- *爱尔朗分布* E_k

  设 $v_1, \cdots, v_k$ 是 k 个相互独立的随机变量，服从相同参数 $k\mu$ 的负指数分布，那么：

$$T = v_1 + v_2 + \cdots + v)k$$

  PDF:

$$b_k(t) = \frac{\mu k (\mu k t)^{k-1}}{(k-1)!} e^{-\mu k t} \quad t > 0$$

1.2 级数展开

基本幂级数

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{1}{n!} x^n, & -\infty < x < +\infty \\ \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}, & -\infty < x < +\infty \\ \frac{1}{1+x} &= \sum_{n=0}^{\infty} (-1)^n x^n, & -1 < x < +1 \end{aligned}$$

- 推广

$$\begin{aligned}
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}, \qquad && -\infty < x < +\infty \\
\frac{1}{1+x^2} &= \sum_{n=0}^{\infty} (-1)^n x^{2n}, && -1 < x < +1 \\
\ln(1+x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1}, && -1 < x < +1 \\
a^x = e^{xlna} &= \sum_{n=0}^{\infty} \frac{(\ln a)^n}{n!} x^n, && -\infty < x < +\infty \\
\arctan x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, && -1 \le x \le +1
\end{aligned}$$

泰勒展开

$$f(x) = \frac{f(x_0)}{0!} + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{n!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

拓展：麦克劳林公式

$$\begin{aligned}
e^x &= 1 + x + \frac{1}{2!}x^2 + \cdots + \frac{1}{n!}x^n + o(x^n) \\
\sin x &= x - \frac{1}{3!}x^3 + \cdots + \frac{(-1)^{m-1}}{(2m-1)!}x^{2m-1} + o(x^{2m-1}) \\
\cos x &= 1 - \frac{1}{2!}x^2 + \cdots + \frac{(-1)^m}{(2m)!}x^{2m} + o(x^{2m}) \\
\ln(1+x) &= x - \frac{1}{2}x^2 + \cdots + \frac{(-1)^{n-1}}{n}x^n + o(x^n)
\end{aligned}$$

佩亚诺余项为 $(x-x_0)^n$ 的高阶无穷小：$R_n(x) = o\left[(x-x_0)^n\right]$

1.2 运行指标

排队系统运行指标间的关系：

- λ：单位时间内顾客的平均到达数，则 $1/\lambda$ 表示向量两个顾客到达的平均时间；

- μ：单位时间内被服务完毕离去的*平均顾客数*，$1/\mu$ 表示对每个顾客的*平均服务时间*
- S：服务系统中并联的服务台数
- $P_n(t)$：时刻 t 系统中恰有 n 个顾客的概率。

排队系统中运行指标之间的关系：

$$
\begin{aligned}
&L_s = \lambda W_s && W_s = \frac{L_s}{\lambda} \\
&L_q = \lambda W_q && W_q = \frac{L_q}{\lambda} \\
&L_s = L_q + \frac{\lambda}{\mu} && W_s = W_q + \frac{1}{\mu} \\
&L_s = \sum_{n=0}^{\infty} nP_n && W_q = W_s - \frac{1}{\mu} = \frac{\rho}{\mu-\lambda} \\
&L_q = \sum_{n=0}^{\infty}(n-s)P_n = \frac{\rho\lambda}{\mu-\lambda}
\end{aligned}
$$

- $M/M/1/\infty/\infty$

$$
\begin{aligned}
P_0 &= 1-\rho \\
P_n &= \rho^n P_0 \\
L_s &= \sum_{n=0}^{\infty} nP_n = \rho(1-\rho)\left(\frac{1}{1-\rho}\right)' = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda} \\
W_s &= \frac{L_s}{\lambda} \\
L_q &= L_s - \frac{1}{\mu} \\
W_q &= \frac{L_1}{\lambda}
\end{aligned}
$$

W_s 的 PDF可以表示为:

$$
f(W_s) = (\mu-\lambda)e^{-(\mu-\lambda)t}
$$

有三种方法求解 $\{p_n\}$：

1. 迭代法 *Iterative Method*

2. 母函数法 *Generating Functions*
3. 线性算子 *Operators*

> *-tion 1.5 for all G/G/1 queues.* *In summary, the full steady-state solution for theM/M/1 system is the geometric probability function* *pn = (1 − ρ)pn (ρ = λ/μ < 1). (3*
> [show annotation]

得到 $\{p_n\}$ 为

$$p_n = (1-\rho)\rho^n$$

> *= (1 − ρ)pn (ρ = λ/μ < 1). (3.9)* *We emphasize that the existence of a steady-state solution depends on the conditionthat ρ < 1, or equivalently, λ < μ .* *This makes intuitive sense, for*
> [show annotation]

> *in comparisonwith other models.* *Finally, we note that for some models, it is relatively easy to find a closed expression for $P(z)$, but quite difficult to find its series expansion to obtain the $\{p_n\}$* *.However, even if the series exp*
> [show annotation]

3.2.4 Measures of Effectiveness

> *.3.2.4 Measures of Effectiveness* *The steady-state probability distribution for the system size allows us to calculatethe system's measures of effectiveness.* *Two of immediate interest are t*
> [show annotation]

- *系统中平均顾客数* L
  可以根据系统的稳定概率分布来计算系统的效益指标。首先考虑当系统处于稳态时，系统中顾客数的期望和队列中顾客数的期望。

设随机变量 N 表示稳态时系统中的顾客数，L 表示其期望，则

$$
\begin{aligned}
L = \mathbb{E}[N] &= \sum_{n=0}^{\infty} n p_n \\
&= (1-\rho) \sum_{n=0}^{\infty} n \rho^n \\
\\
&= \rho(1-\rho) \sum_{n=0}^{\infty} n \rho^{n-1} \\
&= \rho(1-\rho) \left(\frac{1}{1-\rho} \right)' \\
&= \frac{\rho(1-\rho)}{(1-\rho)^2} \\
&= \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}
\end{aligned}
$$

- *平均队列长度* L_q

$$
\begin{aligned}
L_q &= \sum_{n=1}^{\infty} (n-1) p_n \\
&= \sum_{n=1}^{\infty} n p_n - \sum_{n=1}^{\infty} p_n = L_(1-p_0) \\
&= \frac{\rho}{1-\rho} - \rho \\
&= \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)}
\end{aligned}
$$

- *队列不为空时的平均队列长度* L_q'

$$
L_q' = \mathbb{E}[N_q | N_q \neq 0] = \sum_{n=1}^{\infty} (n-1) p_n' = \sum_{n=2}^{\infty} (n-1) p_n'
$$

其中，p_n' 表示队列不为空时的条件下系统的顾客数为n 的条件概率。

$$
\begin{aligned}
p_n' &= \frac{Pr\{n \text{ in system and } n \geq 2\}}{Pr\{n \geq 2\}} \\
&= \frac{p_n}{\sum\limits_{n=2}^{\infty} p_n} \qquad (n \geq 2) \\
&= \frac{p_n}{1 - p_0 - p_1} = \frac{p_n}{1 - (1-\rho) - (1-\rho)\rho} \\
&= \frac{p_n}{\rho^2}
\end{aligned}
$$

Then

$$
\begin{aligned}
L_q' &= \sum_{n=2}^{\infty} (n-1) p_n' \\
&= \sum_{n=2}^{\infty} (n-1) \frac{(1-\rho)\rho^n}{\rho^2} \\
&= (1-\rho) \sum_{n=2}^{\infty} (n-1) \rho^{n-2} \\
&= (1-\rho) \left(\sum_{n=0}^{\infty} n \rho^{n-1} \right) \\
&= (1-\rho) \left(\frac{1}{1-\rho} \right)' \\
&= \frac{1}{1-\rho} = \frac{\mu}{\mu - \lambda}
\end{aligned}
$$

- *顾客在系统中的平均等待时间* W

$$
W = \frac{L}{\lambda} = \frac{\rho}{\lambda(1-\rho)} = \frac{1}{\mu - \lambda}
$$

- *队列中的平均等待时间* W_q

$$
W_q = \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{\rho}{\mu - \lambda}
$$

3.2.5 等待时间的分布

- 等待总时间的分布
  在系统中的总等待时间为服从期望为 $1/(\mu-\lambda)$ 的指数分布随机变量，即：

$$
\begin{aligned}
W(t) &= 1-e^{-(\mu-\lambda)t}, \qquad & t \geq 0 \\
w(t) &= (\mu-\lambda)e^{-(\mu-\lambda)t}, \qquad & t > 0
\end{aligned}
$$

- 排队时间的分布
  可以看作是按照 ρ 的概率服从指数分布，$1-\rho$ 的概率排队时间为 0

$$
W_q(t) = 1-\rho+\rho(1-e^{-(\mu-\lambda)t}) = 1-\rho e^{-(\mu-\lambda)t}
$$

$$