> *Decision is a risk rooted in the courage of being free.*
> *— Paul Tillich*

1. Introduction

> *or the organization as a whole. These kinds ofproblems and the need to find a better way to solve them provided the environment for theemergence of operations research (commonly referred to as OR) .The roots of OR can be traced b*
> show annotation

> *oped by George Dantzig in 1947. Many of thestandard tools of OR, such as linear programming, dynamic programming, queueing theory,and inventory theory, were relatively well developed before the end of the 1950s .A second factor that gave great*
> show annotation

Three main characteristics of OR:

1. The research part of the name means that operations research uses an approach that resembles the way research is conducted in established scientific fields.
2. Still another characteristic of OR is its broad viewpoint.
3. An additional characteristic is that OR frequently attempts to search for a best solution (referred to as an optimal solution) for the model that represents the problem under consideration.

> *HE NATURE OF OPERATIONS RESEARCH As its name implies, operations research involves "research on operations." Thus, opera-tions research is applied to problems that concern how to conduct and coordinate theoperations (i.e., the activities) within an organization. The nature of the*

> *organizat*
> [show annotation]

- *Algorithms and Courseware*

> *nd what makes them soefficient.* *You then will use these algorithms to solve a variety of problems on a computer.The OR Courseware contained on the book's website (www.mhhe.com/hillier) will be akey tool for doing all this* *.One special feature in your OR*
> [show annotation]

> *be akey tool for doing all this.* *One special feature in your OR Courseware is a program called OR Tutor. This pro-gram is intended to be your personal tutor to help you learn the algorithms. It consists ofmany demonstration examples that display and explain the algorithms in action. These"demos" supplement the examples in the book* *.In addition, your OR Courseware*
> [show annotation]

> *lement the examples in the book.* *In addition, your OR Courseware includes a special software package calledInteractive Operations Research Tutorial, or IOR Tutorial for short.* *Implemented inJava, this inno*
> [show annotation]

> *cel (as described in Sec. 20.6).* *After many years, LINDO (and its companion modeling language LINGO) continuesto be a popular OR software package* *. Student versions of LINDO and*
> [show annotation]

2. Overview of the Operations Research Modeling Approach

> *ses of a typical large OR study.* *One way of summarizing the usual (overlapping) phases of an OR study is the following:* *1. Define the problem of interes*
> [show annotation]

2.1 Defining the Problem and Gathering Data

> *onclusions of the study will be. It is difficult to extract a "right"answer from the "wrong" problem! The first thing to recognize is*
> *show annotation*

> *recommendations to manage-ment. Frequently, the report to management will identify a number of alternatives that areparticularly attractive under different assumptions or over a different range of values ofsome policy parameter that can be evaluated only by management (e.g., the trade-offbetween cost and benefits) . Management evaluates the s*
> *show annotation*

> *course that the study is taking. Ascertaining the appropriate objectives is a very important aspect of problem defini-tion. To do this, it is necessary fir*
> *show annotation*

> *ns do not use thisapproach. A number of studies of U.S. corporations have found that management tends toadopt the goal of satisfactory profits, combined with other objectives, instead of focusingon long-run profit maximization. Typically, some of these oth*
> *show annotation*

2.2 Formulating a Mathematical Model

> *hips, and facilitating analysis. Mathematical models are also idealized representations, but they are expressed in termsof mathematical symbols and expressions . Such laws of physics as F = ma*
> *show annotation*

要素：

- decision variables
- objective function
- constraints
- parameters

> *typifies the models used in OR.* *Determining the appropriate values to assign to the parameters of the model (onevalue per parameter) is both a critical and a challenging part of the model-building process* *.In contrast to textbook problem*
> [show annotation]

- Sensitivity anylysis

> *cessity, only a rough estimate.* *Because of the uncer-tainty about the true value of the parameter, it is important to analyze how the solutionderived from the model would change (if at all) if the value assigned to the parameter werechanged to other plausible values.* *This process is referred to*
> [show annotation]

> *hout the remainder ofthis book.* *One particularly important type that is studied in the next several chapters is thelinear programming model, where the mathematical functions appearing in both theobjective function and the constraints are all linear functions.* *In Chap. 3, specific linear*
> [show annotation]

> *ality standards at minimum cost.* *Mathematical models have many advantages over a verbal description of the problem* *.One advantage is that a ma*
> [show annotation]

- *由简入繁*

> *uction of themathematical model.* *In developing the model, a good approach is to begin with a very simple version andthen move in evolutionary fashion toward more elaborate models that more nearly reflectthe complexity of the real problem.* *This process of model enrichmen*
> [show annotation]

> *he problem was beingdefined.* *If there are multiple objectives, their respective measures commonly are thentransformed and combined into a composite measure, called the overall measure of performance* *. This overall*

> measure might be
> show annotation

2.3 Deriving Solutions from the Model

> ss the nature of such solutions. A common theme in OR is the search for an optimal, or best, solution. Indeed, manyprocedures have been
> show annotation

> associated with real problems. However, if the model is well for-mulated and tested, the resulting solution should tend to be a good approximation to anideal course of action for the real problem. Therefore, rather than be delud
> show annotation

> can be obtained by other means. The late Herbert Simon (an eminent management scientist and a Nobel Laureate ineconomics) pointed out that satisficing is much more prevalent than optimizing in actualpractice. In coining the term satisficing as a combination of the words satisfactory andoptimizing , Simon was describing the t
> show annotation

> ering OR leaders, Samuel Eilon, "Optimizing is the sci-ence of the ultimate; satisficing is the art of the feasible. "1OR teams attempt to bring as m
> show annotation

> efits resulting from the study. In recognition of this concept,OR teams occasionally use only heuristic procedures (i.e., intuitively designed proceduresthat do not guarantee an optimal solution) to find a good suboptimal solution. This is mostoften the case when
> show annotation

> he problem would be very large. In recent years, great progress has been made indeveloping efficient and effective metaheuristics that provide both a general structure andstrategy guidelines for designing a specific heuristic procedure to fit a particular kind ofproblem. The use of metaheuristics (the s
> show annotation

> Chap. 14) is continuing to grow. The discussion thus far has implied that an OR study seeks to find only one solution,which may or may not be required to be optimal . In fact, this usually is not t
> show annotation

> ddi-tional analysis is needed. T herefore, postoptimality analysis (analysis done after findingan optimal solution) is a very important part of most OR studies. This analysis also issometi
> show annotation

> part of most OR studies. This analysis also issometimes referred to as what-if analysis because it involves addressing some questionsabout what would happen to the optimal solution if different assumptions are made aboutfuture conditions. These questions often are raise
> show annotation

> sing confidence in its validity. In part, postoptimality analysis involves conducting sensitivity analysis to determinewhich parameters of the model are most critical (the "sensitive parameters") in determin-ing the solution . A common definition of sensiti
> show annotation

> ut changingthe optimal solution. Identifying the sensitive parameters is important, because this identifies the parameterswhose value must be assigned with special care to avoid distorting the output of the model. The value assigned to a par
> show annotation

2.4 Testing the Model

> rect as many flaws as possible. Eventually, after a long succession of improved models, theOR team concludes that the current model now is giving reasonably valid results. Althoughsome minor flaws undoubtedly remain hidden in the model (and may never be detected), themajor flaws have been sufficiently eliminated so that the model now can be reliably used

> *.This process of testing and*
> show annotation

- Model validation

> *model now can be reliably used.* *This process of testing and improving a model to increase its validity is commonlyreferred to as model validation.* *It is difficult to describe how*
> show annotation

- 维度一致

> *el may help to reveal mistakes.* *It is also useful to make sure thatall the mathematical expressions are dimensionally consistent in the units used* *. Additionalinsight into the val*
> show annotation

- 回溯检验

> *PREPARING TO APPLY THE MODEL 19* *A more systematic approach to testing the model is to use a retrospective test.* *Whenit is applicable, this t*
> show annotation

> *alternativecourses of actions.* *On the other hand, a disadvantage of retrospective testing is that it uses the same datathat guided the formulation of the model.* *The crucial question is whether*
> show annotation

2.5 preparing to Apply the Model

> *nal PDF to printerother cases,* *an interactive computer-based system called a decision support system isinstalled to help managers use data and models to support (rather than replace) their decisionmaking as needed* *. Another program may generate m*
> show annotation

> *ning OR study.)■ 2.7 CONCLUSIONS* *Although the remainder of this book focuses primarily on constructing and solving mathe-matical models, in this chapter we have tried to emphasize that this constitutes only a por-tion of the overall process involved in conducting a typical OR study* *. The other phasesdescribed h*
> show annotation

3. Introduction to Linear Programming

> *ork through subsequent examples.* *However, a verbal summary may help provide perspective. Briefly, the most common typeof application involves the general problem of allocating limited resources amongcompeting activities in a best possible (i.e., optimal) way* *. More precisely, this proble*
> show annotation

> *describe the problem of concern.* *The adjective linear means that all the mathematical functions in this model are required to be linear functions* *. The word programming does not*
> show annotation

> *required tobe linear functions.* *The word programming does not refer here to computer programming;rather, it is essentially a synonym for planning* *. Thus, linear programming inv*
> show annotation

4. Solving Linear Programming Problems: The Simplex Method

标准形式的线性规划问题：

1. *Maximization* 目标函数最大化
2. 所有约束条件以 $\leq$ 连接
3. *Non-negativity* 所有变量取值为非负的形式；
4. 所有约束右端项 b_i 的值要求为非负

> *solution is an optimal solution. Optimality test: Consider any linear programming problem that possesses atleast one optimal solution. If a CPF solution has no adjacent CPF solutions thatare better (as measured by Z), then it must be an optimal solution. Thus, for the example, (2, 6)*
> show annotation

Transforming to Standard Form:

1. "minimize" objective can be changed to "maximize" by multiplying "-1."
2. All variables can be moved to the LHS (don't forget to change sign).
3. All constants can be moved to the RHS (don't forget to change sign).
4. "$\geq$" constraints can be changed to "≤" by multiplying "-1" on both sides.
5. What about "=" constraints?
   "$=$" constraints can be changed to "$\leq$" by replacing it by two constraints, one "$\geq$" and one "$\leq$". The "$\geq$" is then transformed into "$\leq$" one by multiplying "–1" on both sides.

e.g.

$$x_1 + x_2 = 7$$

is changed to

$$\begin{aligned} x_1 + x_2 \geq 7 \qquad & x_1 + x_2 \leq 7 \\ -x_1 - x_2 \leq -7 \qquad & x_1 + x_2 \leq 7 \end{aligned}$$

> *of this constraint boundary. Ademonstration of these properties is provided by the demonstration example in your ORTutor entitled Interpretation of the Slack Variables .The terminology used in Section*
> show annotation

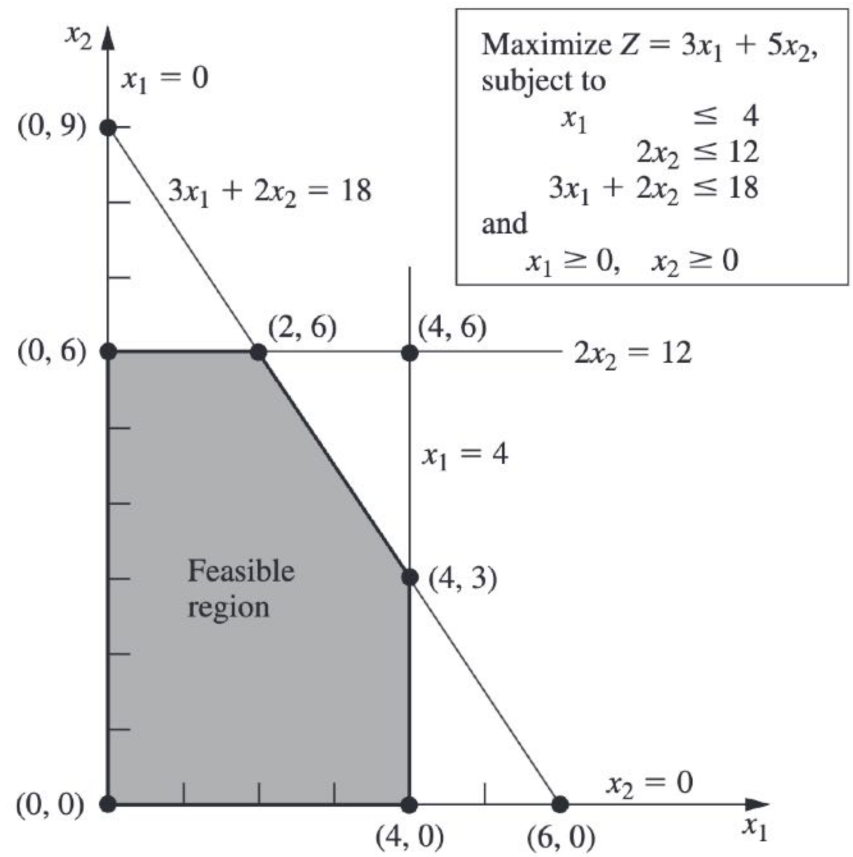

FIGURE 4.1
Constraint boundaries and corner-point solutions for the Wyndor Glass Co. problem.

Corresponding terminology for the augmented form:

- *Augmented solution* (拓展解): Solution for the original decision variables augmented by the slack variables
  - Example: augmenting solution (2,6) yields the augmented solution (2,6,2,0,0)
- *Basic solution* (基本解): Augmented corner-point solution
  - Example: augmenting corner-point solution (4,6) yields basic solution(4,6,0,0,-6)
- *Basic feasible (BF) solution* (基本可行解): Augmented CPF solution
  - Example: the CPF (0,6) is equivalent to the BF solution (0,6,4,0,6)

Note: CPF (and BF) solution can be either *feasible* feasible or *infeasible*. 基本可行解（BF solution）是拓展的 CPF 解。

> *r the problem in augmented form.* ***The only difference between basic solutions and corner-point solutions (or betweenBF solutions and CPF***

> *solutions) is whether the values of the slack variables are included .4.2 SETTING UP THE SIMPLEX METH*
> *show annotation*
>
> *唯一区别在于是否包括松弛变量*

> *number of equations 5 3 2. This fact gives us 2 degrees of freedom in solving the system, since any two variables can bechosen to be set equal to any arbitrary value in order to solve the three equations in terms ofthe remaining three variables .5 The simplex method uses zero*
> *show annotation*

> *e following general definitions. A basic solution has the following properties: 1. Each variable is designated a*
> *show annotation*

方程结构：
基变量个数=约束条件数
非基变量个数（自由度） = 变量总数-约束条件数

一个基本解具有如下性质：

1. 每个变量都可以作为非基变量或基变量
2. 基变量的个数等于约束条件数。因此 *非基变量数* = 变量总数减去约束条件个数
3. 非基变量的值设为 0
4. 基变量的值作为方程组的联立解被求得。
5. 如果基变量满足非负约束，基本解为 *BF 解*

Negative RHS and "≥" Constraints

- **Negative RHS:** $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le -b_1$
  ➔ **Multiply both sizes by -1:** $-a_{11}x_1 - a_{12}x_2 - \ldots - a_{1n}x_n \ge b_1$

- **Constraints with "≥":** $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \ge b_1$
  ➔ **Add both surplus and artificial variables:**
  1) Surplus : $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n - x_s = b_1$ where $x_s \ge 0$ (surplus variable)
  2) Artificial: $a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n - x_s + x_a = b_1$ where $x_a \ge 0$ (artificial variable)
  Add $-M^*x_a$ to objective function

- The main purpose of adding artificial variables is to get a starting point for Simplex.

- **Minimization:** Minimize $Z = c_1x_1 + c_2x_2 + \ldots + c_nx_n$
  ➔ **Multiply objective by "–1":** **Maximize $-Z$** $= -c_1x_1 - c_2x_2 - \ldots - c_nx_n$

求解等号或大于约束：

- Big M method

Iteration	Basic variables	Eq.	Coefficient of							Right Side	Solution optimal?
			Z	x1	x2	x3	x4	x5	x6		
0	Z	(0)	-1	-5M+3	-4M+2	-8M+4	0	M	0	-180M	No
	x4	(1)	0	2	1	3	1	0	0	60	
	x6	(3)	0	3	3	5	0	-1	1	120	
1	Z	(0)	-1	1/3(M+1)	-1/3(4M-2)	0	1/3(8M-4)	M	0	-20M-80	No
	x3	(1)	0	2/3	1/3	1	1/3	0	0	20	
	x6	(3)	0	- 1/3	4/3	0	-5/3	-1	1	20	
2	Z	(0)	-1	1/2	0	0	M-1/2	-1/2	M-1/2	-90	Yes
	x3	(1)	0	3/4	0	1	3/4	1/4	- 1/4	15	
	x2	(3)	0	- 1/4	1	0	-5/4	- 3/4	3/4	15	

- Two phase method

Back to Radiation Therapy Example: Two-Phase Method

- Two-phase method
  - Phase 1: minimize $Z = \bar{x}_4 + \bar{x}_6$ (until $\bar{x}_4 = 0, \bar{x}_6 = 0$)
    - Obtain a BF solution for the real problem. This solution is used as the initial BF solution to real problem in phase 2.
  - Phase 2: minimize $Z = 0.4x_1 + 0.5x_2$ (with $\bar{x}_4 = 0, \bar{x}_6 = 0$)

Phase 1 Problem (Radiation Therapy Example):

Minimize $Z = \bar{x}_4 + \bar{x}_6$,

subject to

$$\begin{aligned} 0.3x_1 + 0.1x_2 + x_3 &= 2.7 \\ 0.5x_1 + 0.5x_2 + \bar{x}_4 &= 6 \\ 0.6x_1 + 0.4x_2 - x_5 + \bar{x}_6 &= 6 \end{aligned}$$

and

$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad \bar{x}_4 \geq 0, \quad x_5 \geq 0, \quad \bar{x}_6 \geq 0.$

Phase 2 Problem (Radiation Therapy Example):

Minimize $Z = 0.4x_1 + 0.5x_2$,

subject to

$$\begin{aligned} 0.3x_1 + 0.1x_2 + x_3 &= 2.7 \\ 0.5x_1 + 0.5x_2 &= 6 \\ 0.6x_1 + 0.4x_2 - x_5 &= 6 \end{aligned}$$

and

$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0, \quad x_5 \geq 0.$

Phase 1											
Iteration	Basic variables	Eq.	Coefficient of							Right Side	Solution optimal?
			Z	x1	x2	x3	x4	x5	x6		
0	Z	(0)	-1	-5	-4	-8	0	1	0	-180	No
	x4	(1)	0	2	1	3	1	0	0	60	
	x6	(3)	0	3	3	5	0	-1	1	120	
1	Z	(0)	-1	1/3	-4/3	0	8/3	1	0	-20	No
	x3	(1)	0	2/3	1/3	1	1/3	0	0	20	
	x6	(3)	0	- 1/3	4/3	0	-5/3	-1	1	20	
2	Z	(0)	-1	0	0	0	1	0	1	0	Yes
	x3	(1)	0	3/4	0	1	3/4	1/4	- 1/4	15	
	x2	(3)	0	- 1/4	1	0	-5/4	- 3/4	3/4	15	
Phase 2											
3	Z	(0)	-1	3	2	4		0		0	No
	x3	(1)	0	3/4	0	1		1/4		15	
	x2	(3)	0	- 1/4	1	0		- 3/4		15	
4	Z	(0)	-1	1/2	0	0		1/2		-90	Yes
	x3	(1)	0	3/4	0	1		1/4		15	
	x2	(3)	0	- 1/4	1	0		- 3/4		15	

5. Theory of the Simple Method

5.1 Foundations of the Simplex Method

> *solution in n-dimensional space. A corner-point feasible (CPF) solution is a feasible solution that does not lieon any line segment1 connecting two other*

> feasible solutio ns.As this definition implies, a
> show annotation

> his edge of the feasible region. Properties of CPF Solutions We now focus on three key
> show annotation

CPF 解的性质

1. 1）如果只有一个是最优解，他一定是 CPF 解；2）如果有许多最优解（在有界可行域重），至少两个必为相邻 CPF 解
2. 只有有限个 CPF 解
3. 如果一个 CPF 解没有相邻 CPF 解比它更优（以 Z 来衡量），那么就不存在任何更好的 CPF 解。这样，假定这个问题至少有一个最优解（由问题具有可行解和一个有界的可行鱼来保证），这个 CPF 解就是最优解（性质一）。

5.2 The Simplex Method in Matrix Form

Using matrices, the standard form can be given by:

$$\max Z = cX$$

subjective to

$$Ax \leq b \quad \text{and} \quad x \geq 0,$$

Matrix Form of the Current Set Functions

- The original set of equations are $$\begin{bmatrix}1 & -\mathbf{c} & \mathbf{0}\\ \mathbf{0} & \mathbf{A} & \mathbf{I}\end{bmatrix}\begin{bmatrix}Z\\ \mathbf{x}\\ \mathbf{x}_s\end{bmatrix}=\begin{bmatrix}0\\ \mathbf{b}\end{bmatrix}$$
- The algebraic operation performed by simplex method is captured in the matrix form.

$$\begin{bmatrix}Z\\ \mathbf{x}_B\end{bmatrix}=\begin{bmatrix}1 & \mathbf{c}_B\mathbf{B}^{-1}\\ \mathbf{0} & \mathbf{B}^{-1}\end{bmatrix}\begin{bmatrix}0\\ \mathbf{b}\end{bmatrix}=\begin{bmatrix}\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}\\ \mathbf{B}^{-1}\mathbf{b}\end{bmatrix}$$

- Use above matrix that premultiplies the original right-hand side to premultiply the original left-hand side.

$$\begin{bmatrix}1 & \mathbf{c}_B\mathbf{B}^{-1}\\ \mathbf{0} & \mathbf{B}^{-1}\end{bmatrix}\begin{bmatrix}1 & -\mathbf{c} & \mathbf{0}\\ \mathbf{0} & \mathbf{A} & \mathbf{I}\end{bmatrix}=\begin{bmatrix}1 & \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c} & \mathbf{c}_B\mathbf{B}^{-1}\\ \mathbf{0} & \mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1}\end{bmatrix}$$

- The set of equation after any iteration is

$$\begin{bmatrix}1 & \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A}-\mathbf{c} & \mathbf{c}_B\mathbf{B}^{-1}\\ \mathbf{0} & \mathbf{B}^{-1}\mathbf{A} & \mathbf{B}^{-1}\end{bmatrix}\begin{bmatrix}Z\\ \mathbf{x}\\ \mathbf{x}_s\end{bmatrix}=\begin{bmatrix}\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b}\\ \mathbf{B}^{-1}\mathbf{b}\end{bmatrix}$$

Summary of the Matrix form of the simples solution

- *Initialization*: Introduce slack variable to obtain initial basic variable. This yields initial $x B$, $c B$,B .
- *Iteration*
  Step 1: Determine the entering basic variable
  - Check coefficient of nonbasic variable in Eq(0)
  - Select negative coefficient having the largest value
  Step 2. Determine the leaving basic variable
  - Use $B^{-1}A, x_B = B^{-1}b$ and minimal ratio
  Step 3. Determine the new BF solution
  - Update B, x_B, c_B
  Optimality test: The solution is optimal iff positive $c_B B^{-1}A - c$ and $c_B B^{-1}$ are positive.

E.g.

Wyndor Glass Co. Example

- The input parameters are

$$\mathbf{c} = [3, 5], \quad [\mathbf{A}, \mathbf{I}] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}$$

- Initialization

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}, \quad \mathbf{c}_B = [0, 0, 0], \quad \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{B}^{-1}$$

- Optimality test

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} = [0, 0] - [3, 5] = [-3, -5]$$

not optimal, above is the coefficient of nonbasic varialbes x_1 and x_2

> *at the end of this section.* ***These appli-cations are particularly important only when we are dealing with the final simplex tableauafter the optimal solution has been obtained*** *. Therefore, we will focus h*
>
> show annotation

Let $\mathbf{B}$ be the basis matrix for the *optimal* solution

- $\mathbf{S}^* = \mathbf{B}^{-1}$ = coefficient of the *slack* variables in rows 1 to m
- $\mathbf{A}^* = \mathbf{B}^{-1}\mathbf{A}$ = coefficient of the *original* variables in rows 1 to m
- $\mathbf{y}^* = \mathbf{c_B}\mathbf{B}^{-1}$ = coefficient of the *slack* variables in rows 0
- $\mathbf{z}^* = \mathbf{c_B}\mathbf{B}^{-1}\mathbf{A}$, so $\boldsymbol{z}^* - \boldsymbol{c}$ is coefficient of the *original* variables in rows 0
- $\mathrm{Z}^* = \mathbf{c_B}\mathbf{B}^{-1}\mathbf{b}$ = optimal value of the objective function
- $\mathbf{b}^* = \mathbf{B}^{-1}\mathbf{b}$ = optimal right-hand sides of rows 1 to m

- Suppose initial tableau $\mathbf{t}$ and $\mathbf{T}$ are given, and just $\boldsymbol{y}^*$ and $\boldsymbol{S}^*$ are given. How to use this information to calculate the final tableau?
  - $\mathbf{t} = [-\mathbf{c}, \mathbf{0}, 0]$, $\mathbf{T} = [\mathbf{A}, \mathbf{I}, \mathbf{b}]$

设 B 是用单纯形法找出最优解时的基矩阵，令

- $S^* = B^{-1}$ 表示第一行至 m 行中 *松弛变量* 的系数
- $A^* = B^{-1}A$ 表示第一行至 m 行中 *原变量* 的系数
- $y^* = c_B B^{-1}$ 表示第 0 行中 *松弛变量* 的系数
- $z^* = c_B B^{-1} A$，所以 $z^* - c = 0$ 为第 0 行中*原变量*的系数
- $Z^* = c_B B^{-1} b$ 表示目标函数的最优值
- $b^* = B^{-1} b$ 表示第一行至 m 行最优的右端项值

5.4 The Revised Simplex Method

$$\left(\mathbf{B}_{\text{new}}^{-1}\right)_{ij} = \begin{cases} \left(\mathbf{B}_{\text{old}}^{-1}\right)_{ij} - \frac{a'_{ik}}{a'_{rk}}\left(\mathbf{B}_{\text{old}}^{-1}\right)_{rj} & \text{if } i \neq r \\ \frac{1}{a'_{rk}}\left(\mathbf{B}_{\text{old}}^{-1}\right)_{rj} & \text{if } i = r. \end{cases}$$

$$\mathbf{c} = [3, 5], \quad [\mathbf{A}, \mathbf{I}] = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 12 \\ 18 \end{bmatrix}.$$

Wyndor Glass Co. Example

- Iteration 1

$$\boldsymbol{\eta} = \begin{bmatrix} -\frac{a_{12}}{a_{22}} \\ \frac{1}{a_{22}} \\ -\frac{a_{32}}{a_{22}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ -1 \end{bmatrix} \qquad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

- Iteration 2

$$\boldsymbol{\eta} = \begin{bmatrix} -\frac{a'_{11}}{a'_{31}} \\ -\frac{a'_{21}}{a'_{31}} \\ \frac{1}{a'_{31}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix} \qquad \mathbf{B}^{-1} = \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

B^{-1} _{now} E B^{-1} _{old}

6. Duality theory

> s many important ramifications. This discovery re-vealed that every linear programming problem has associated with it another linear pro-gramming problem called the dual. The relationships between the d
>
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> amming problem called the dual. The relationships between the dual problem and theoriginal problem (called the primal) prove to be extremely useful in a variety of ways .For example, you soon will see
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6.1 The Essence of Duality Theory

> HE ESSENCE OF DUALITY THEORY Given our standard form for the primal problem at the left (perhaps after conversion fromanother form), its dual problem has the form shown to the right. hil23453_ch06_197-224.qxd 1/15/
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Primal Problem

Maximize $Z = \sum_{j=1}^{n} c_j x_j,$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \qquad \text{for } i = 1, 2, \ldots, m$$

and

$$x_j \ge 0, \qquad \text{for } j = 1, 2, \ldots, n.$$

Dual Problem

Minimize $W = \sum_{i=1}^{m} b_i y_i,$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \ge c_j, \qquad \text{for } j = 1, 2, \ldots, n$$

and

$$y_i \ge 0, \qquad \text{for } i = 1, 2, \ldots, m.$$

原问题可以理解为最大利润，对偶问题为最小化成本，所以的得到影子价格，构成目标函数的参数。

> M Page 202 Final PDF to printer Summary of Primal-Dual Relationships Now let us summarize the newly d

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Summary of Primal-Dual Relationships

1. **Weak duality property**: If x is a *feasible solution* for the primal problem and y is a feasible solution for the dual problem, then $cx \leq yb$. One application of the dual is to provide a bound for the optimal value.
2. **Strong duality property**: If x^* is an *optimal solution* for the primal problem and y *is an optimal solution for the dual problem, then $cx^* = y^*b
   . 这说明，当两个问题中有一个不是最优解，或者两个都不是最优解时，会有$cx
   . 且当两个都是最优解的时候（目标函数相等），则是 $cx = yb$
3. **Complementary solutions property**: At each iteration, it simultaneously identifies a *CPF solution* x for the primal problem and a complementary solution (互补解) y for the dual problem, and $cx = yb$
4. **Complementary optimal solutions property**: At the final iteration, it simultaneously identifies a *optimal solution* x^* for the primal problem and a complementary optimal solution (最优互补解) y for the dual problem, and $cx^* = y^*b$
5. **Symmetry property:** 对于任意一个远问题和他的对偶问题，两个问题之间的一切关系必定是对称的。
6. **Duality Property**: The following are the only possible relationships between the primal and dual problems.
   - 如果一个问题拥有可行解和有边界的目标函数（存在最优解），那么另一个问题也会有可行解和有界的目标函数。If one problem has feasible solutions and a bounded objective function (and so has an optimal solution), then so does the other problem, so both the weak and strong duality properties are applicable.
   - 如果一个问题拥有可行解，但是目标函数是无界的（无最优解），那么另一个问题没有可行解。If one problem has feasible solutions and an unbounded objective function (and so no optimal solution), then the other problem has no feasible solutions.
   - 如果一个问题没有可行解，那么另一个问题或者没有可行解，或者有可行解，但是目标函数无界。If one problem has no feasible solutions, then

the other problem has either no feasible solutions or an unbounded objective function

- Duality theorem identifies the only possible relationships between the primal and dual problems
  - If one is feasible and bounded, the other must be feasible and bounded.
  - If one is feasible but unbounded, the other must be infeasible.
  - If one is infeasible, the other must be infeasible or unbounded.

		Dual		
		Infeasible	Optimal	Unbounded
Primal	Infeasible	✓	×	✓
	Optimal	×	✓	×
	Unbounded	✓	×	×

Note: Feasible refers to feasible zone, *bounded* refers to objective function

6.3 Prime-Dual Relationship

Complementary slackness property

- **Complementary slackness property**
  - The variables in the primal basic solution and the complementary dual basic solution satisfy the complementary slackness relation.

TABLE 6.8 Complementary slackness relationship for complementary basic solutions

Primal Variable	Associated Dual Variable	
Basic	Nonbasic	(m variables)
Nonbasic	Basic	(n variables)

  - For each pair of associated variable, if one of them has slack in the nonnegativity constrain (a basic variable > 0), then the other one must have no slack (a nonbasic variable = 0).

> *com-plementary basic solution. Complementary slackness property: Given the association between variablesin Table 6.7, the variables in the primal basic solution and the complementarydual basic solution satisfy the complementary slackness relationship shown inTable 6.8. Furthermore, this*

> relationship is a symmetric one, so that these twobasic solutions are complementary to each other. 6.3 PRIMAL-DUAL RELATIONSHIPS 20
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6.4 Adapting to Other Primal Forms

- Sensible-odd-bizarre (大部分、小部分、极个别方法)

> er prob-lem, and vice versa. The sensible-odd-bizarre method, or SOB method for short, saysthat the form of a functional constraint or the constraint on a variable in the dual problemshould be sensible, odd, or bizarre , depending on whether the form
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TABLE 6.14 Corresponding primal-dual forms

Label	Primal Problem (or Dual Problem)		Dual Problem (or Primal Problem)
	Maximize Z (or W)		Minimize W (or Z)
	Constraint i:		Variable y_i (or x_i):
Sensible	$\le$ form	⟷	$y_i \ge 0$
Odd	$=$ form	⟷	Unconstrained
Bizarre	$\ge$ form	⟷	$y_i' \le 0$
	Variable x_j (or y_j):		Constraint j:
Sensible	$x_j \ge 0$	⟷	$\ge$ form
Odd	Unconstrained	⟷	$=$ form
Bizarre	$x_j' \le 0$	⟷	$\le$ form

表 6.14 原问题—对偶问题对应的形式

标　　签	原问题(或者对偶问题)		对偶问题(或者原问题)
	max Z(或者 W)		min W(或者 Z)
	第 i 个约束方程		变量 y_i(或者变量 x_i)
大部分(S)	≤的形式	⟷	$y_i \geq 0$
小部分(O)	=的形式	⟷	无约束
极个别(B)	≥的形式	⟷	$y_i' \leq 0$
	变量 x_j(或者变量 y_j)		第 j 个约束方程
大部分(S)	$x_j \geq 0$	⟷	≥的形式
小部分(O)	无约束	⟷	=的形式
极个别(B)	$x_j' \leq 0$	⟷	≤的形式

6.7 Sensitive Analysis

> [【运筹学】-对偶理论与灵敏度分析(三)(灵敏度分析)_哔哩哔哩_bilibili]()

问题： a_{ij}、b_i、c_j 系数中有一个或几个变化时，原线性规划最优解会有什么变化？

方法： 把发生变化的系数经过一定计算代入原最终表中，进行检查和分析

原问题	对偶问题	结论或继续计算的步骤
可行解	可行解	表中的解仍为最优解
可行解	非可行解	用单纯形法继续迭代求最优解
非可行解	可行解	用对偶单纯形法继续迭代求最优解
非可行解	非可行解	引入人工变量，编制新的单纯形表，求最优解

资源限量 b_i 的灵敏度分析

（1）求 b_1 的变化范围，使最优基不变

要使最有基不变，则 $X_B' = B^{-1}(b+\Delta b) \geq 0$

设 b_1 的变化量为 Δb_1，则有

$$X_B' = B^{-1}b + B^{-1}\Delta b$$

$$= \begin{bmatrix} 5 \\ 5 \\ 15 \end{bmatrix} + \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta b_1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 5 \\ 15 \end{bmatrix} + \begin{bmatrix} \Delta b_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5+\Delta b_1 \\ 5 \\ 15 \end{bmatrix} \geq 0$$

求解得 $\Delta b_1 \geq -5$，所以 b_1 的取值范围为 $[35, +\infty]$

		c_j	1	1	3	0	0	0
C_B	X_B	b	x_1	x_2	x_3	x_4	x_5	x_6
0	x_4	5	0	-2	0	1	-1	-1
1	x_1	5	1	1	0	0	1	-1
3	x_3	15	0	1	1	0	0	1
		σ_j	0	-3	0	0	-1	-2

13 nonlinear Programming

Types of Nonliear Programming Problems

- Unconstrained optimization
  - Minimize $f(x)$ for all values of $x=(x_1,x_2,\cdots,x_n)$
  - No constraints
- Linearly constrained optimization
  - All constraint functions are linear
  - Objective function is nonlinear
- Quadratic programming
  - Linear constraints
  - Objective function is quadratic and convex
- Separable programming
  Each term involves just a single variable
  The function is separable into a sum of functions of individual variables

$$f(x)=\sum_{j=1}^{n} f_j(x_j)$$

- Nonconvex programming
  Functions do not satisfy assumptions of convex programming
  No algorithm will find an optimal solution for all such problems
  Some algorithms explore various parts of the feasible region and may find a global maximum
- Convex programming
  $f(x)$ and $g_i(x)$ are convex functions
- Geometric programming
- Fractional programming

convex function: A function $f: R^n \to R$ is convex if dom f is a convex set and if for all $x,y \in domf$ and θ with $0 \leq \theta \leq 1$, we have:

$$f(\theta x+(1-\theta)y \leq \theta f(x)+(1-\theta)f(y)$$

函数为凹函数,则二阶导数大于0;函数为凸函数,则二阶导数小于零.

First-order condition

Assume f is differentiable (i.e., its gradient ∇f exists at each point in $\mathbf{dom} f$, which is open). Then f is convex if and only if $\mathbf{dom} f$ is convex and

$$f(y) \geq f(x) + \nabla f(x)^T (y - x)$$

holds for all $x, y \in \mathbf{dom} f$.

Second-order condition

Assume f is twice differentiable, that is, its Hessian or second derivative $\nabla^2 f$ exists at each point in $\mathbf{dom} f$, which is open. Then f is convex if and only if $\mathbf{dom} f$ is convex and its Hessian is positive semidefinite.

Operation that preserve convexity

Example of convex function

- Exponential. e^{ax} is convex on $\mathbf{R}$, for any $a \in \mathbf{R}$.
- Powers. x^a is convex on $\mathbf{R}_+$ when $a \geq 1$ or $a \leq 0$, and concave for $0 \leq a \leq 1$.
- Powers of absolute value. $|x|^p$, for $p \geq 1$, is convex on $\mathbf{R}$.
- Logarithm. $\log x$ is concave on $\mathbf{R}_+$.

Example of convex function

- Quadratic-over-linear function. The function $f(x, y) = x^2/y$, with $\mathbf{dom} f = R \times R_+ = \{(x, y) \in \mathbf{R}^2 | y > 0\}$, is convex.
- Log-sum-exp. The function $f(x) = log(e^{x_1} + e^{x_2})$ is convex on $\mathbf{R}^2$.

- *Nonnegative weighted sums*

$$f = w_1 f_1 + \cdots + w_m f_m$$

- *Pointwise maximum and supremum*
  If f_1 and f_2 are convex functions then their pointwise maximum f, defined by $f(x) = \max\{f_1(x), f_2(x)\}$, is also convex
- *Composition*

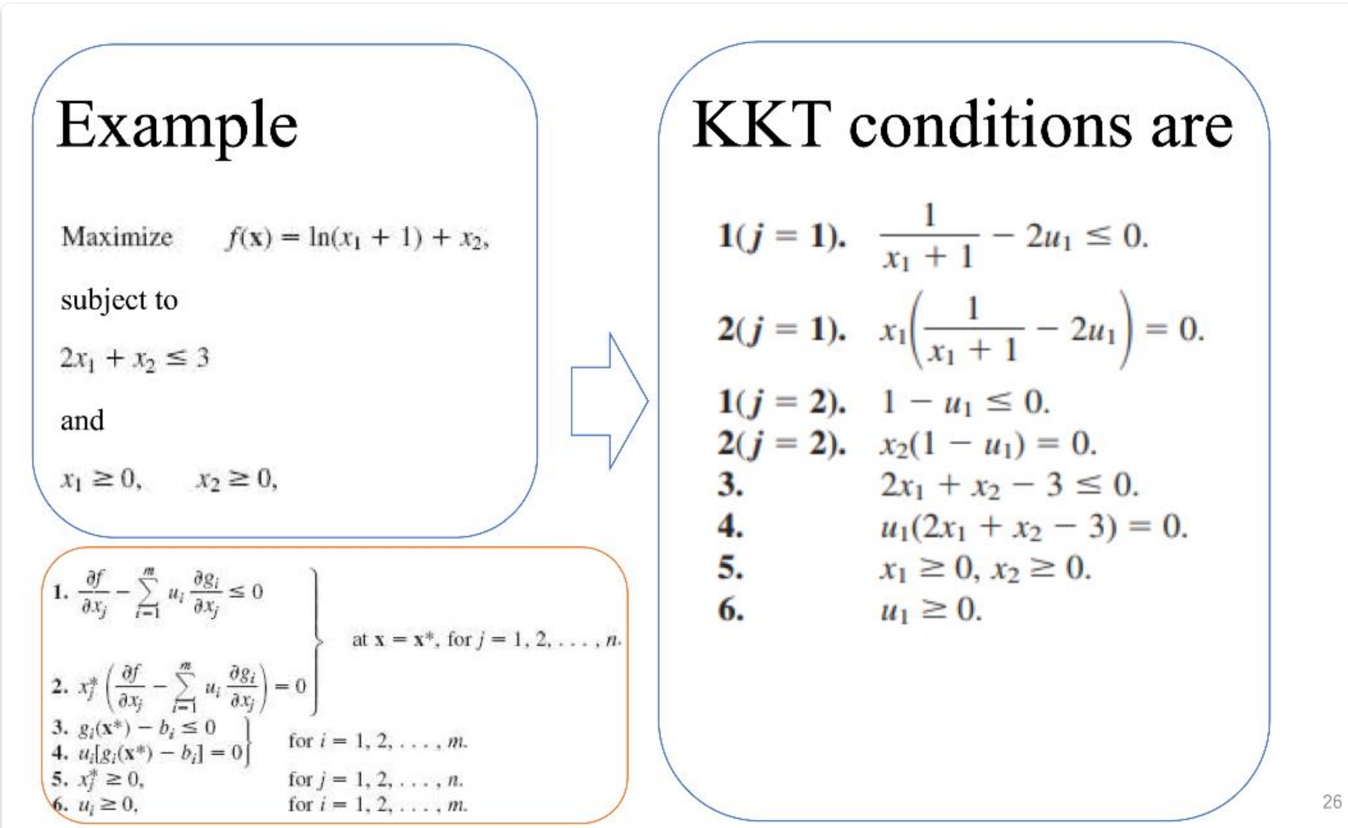
- Composition.
  Assume $h : \mathbf{R} \to \mathbf{R}$, $g : \mathbf{R}^n \to \mathbf{R}$, and $f = h \circ g : \mathbf{R}^n \to \mathbf{R}$, defined by

$$f(x) = h(g(x)), \quad \mathbf{dom} f = \{x \in \mathbf{dom} g \mid g(x) \in \mathbf{dom} h\}.$$

  - f is convex if h is convex and nondecreasing, and g is convex,
  - f is convex if h is convex and nonincreasing, and g is concave,
  - f is concave if h is concave and nondecreasing, and g is concave,
  - f is concave if h is concave and nonincreasing, and g is convex.

> *een being negative and positive.* ***CONVEX OR CONCAVE FUNCTIONS OF SEVERAL VARIABLES*** *The concept of a convex or conca*
> [show annotation]

KKT condition



E.g.
Ex. 5
Consider the following convex programming problem:

$$\text{Maximize} \qquad f(x) = 24x_1 - x_1^2 + 10x_2 - x_2^2$$

subject to

$$x_1 \le 10$$
$$x_2 \le 15$$

and
$x_1 \ge 0, \quad x_2 \ge 0$

1. Use the KKT conditions for this problem to derive an optimal solution.

Solution for 5a

Let

$$\begin{aligned} F(x) &= f(x) - \alpha g(x_1) - \beta g(x_2) \\ &= 24x_1 - x_1^2 + 10x_2 - x_2^2 - \alpha(x_1 - 10) - \beta(x_2 - 15) \end{aligned}$$

where $\alpha, \beta >\geq 0$. According to the KKT theorem, for the problem, we can get the following conditions:

$$\begin{aligned} \frac{\partial F(x)}{\partial x_1} = 24 - 2x_1 - \alpha \leq 0 \\ \frac{\partial F(x)}{\partial x_2} = 10 - 2x_2 - \beta \leq 0 \\ x_1 \frac{\partial F(x)}{\partial x_1} = x_1(24 - 2x_1 - \alpha) = 0 \\ x_2 \frac{\partial F(x)}{\partial x_2} = x_2(10 - 2x_2 - \beta) = 0 \\ \alpha(x_1 - 10) = 0 \\ \beta(x_2 - 15) = 0 \\ x_1 - 10 \leq 0 \\ x_2 - 15 \leq 0 \\ x_1, x_2, \alpha, \beta \geq 0 \end{aligned}$$

Then, we can solve the above system, get the following result:

$$\begin{aligned} x_1 = 10 \\ x_2 = 5 \\ \alpha = 4 \\ \beta = 0 \end{aligned}$$

Thus, the optimal maximum results is $f(x) = 24 \times 10 - 10^2 + 10 \times 5 - 5^2 = 165$